DSCC/CSC 435 & ECE 412 Optimization for Machine Learning — Lecture 6

Proximal Gradient Method

Lecturer: Jiaming Liang — *September 26, 2023*

1 Proximal operator

Definition 1. *Given a function f, the proximal mapping of f is given by*

$$\operatorname{prox}_f(x) = \operatorname{argmin}_{u\in\mathbb{R}^n}\left\{f(u)+\frac{1}{2}\|u-x\|^2\right\},\quad \forall x\in\mathbb{R}^n.$$

Note that if f is closed and convex then $\operatorname{prox}_f(x)$ is a singleton for any $x\in\mathbb{R}^n$.

Example: soft-thresholding, for some $\lambda>0$, the proximal mapping for the one-dimensional function $\lambda|\cdot|$ is

$$\operatorname{prox}_{\lambda|\cdot|}(y) = \mathcal{T}_\lambda(y) = [|y|-\lambda]_+\operatorname{sgn}(y) = \begin{cases} y-\lambda, & y\ge\lambda\\ 0, & |y|<\lambda\\ y+\lambda, & y\le-\lambda\end{cases}$$

Hence, the proximal mapping for $f(x)=\lambda\|x\|_1$ is

$$\mathcal{T}_\lambda(x)\equiv(\mathcal{T}_\lambda(x_j))_{j=1}^n = [|x|-\lambda\mathbf{1}]_+\odot\operatorname{sgn}(x)$$

where $\odot$ denotes componentwise multiplication.

Theorem 1. *Let $Q\subseteq\mathbb{R}^n$ be nonempty. Then, $\operatorname{prox}_{I_Q}(x)=\operatorname{proj}_Q(x)$ for any $x\in\mathbb{R}^n$. Let $Q\subseteq\mathbb{R}^n$ be a nonempty closed convex set. Then, $\operatorname{prox}_{I_Q}(x)=\operatorname{proj}_Q(x)$ is a singleton for any $x\in\mathbb{R}^n$.*

Theorem 2. *Let f be a closed and convex function. Then for any $x,y\in\mathbb{R}^n$, we have*

(i) $\|\operatorname{prox}_f(x)-\operatorname{prox}_f(y)\|^2\le\langle\operatorname{prox}_f(x)-\operatorname{prox}_f(y),x-y\rangle$;

(ii) $\|\operatorname{prox}_f(x)-\operatorname{prox}_f(y)\|\le\|x-y\|$.

Proof. (a) Let $u=\operatorname{prox}_f(x)$ and $v=\operatorname{prox}_f(y)$. It follows from the defintion of proximal mapping that

$$u=\operatorname{argmin}_{w\in\mathbb{R}^n}\left\{f(w)+\frac{1}{2}\|w-x\|^2\right\}$$

and

$$x-u\in\partial f(u).$$

The inclusion is equivalent to

$$f(w) \geq f(u) + \langle x - u, w - u \rangle \quad \forall w \in \mathbb{R}^n.$$

Taking $w = v$, we have

$$f(v) \geq f(u) + \langle x - u, v - u \rangle.$$

Following the same argument for $v = \mathrm{prox}_f(y)$, we have

$$f(u) \geq f(v) + \langle y - v, u - v \rangle$$

Adding the above two inequalities, we obtain

$$0 \geq \langle y - x + u - v, u - v \rangle,$$

i.e.,

$$\langle x - y, u - v \rangle \geq \|u - v\|^2.$$

Plugging $u = \mathrm{prox}_f(x)$ and $v = \mathrm{prox}_f(y)$ into the above inequality, we prove (a).

(b) This statement simply follows from (a) using the Cauchy-Schwarz inequality. □

2 Moreau envelope

Theorem 3. *(**Moreau decomposition**) Let f be a closed and convex function. Then for any $x \in \mathbb{R}^n$, we have*

$$\mathrm{prox}_f(x) + \mathrm{prox}_{f^*}(x) = x.$$

Proof. Let $u = \mathrm{prox}_f(x)$. It is equivalent to $x - u \in \partial f(u)$. Using Theorem 2 of Lecture 5, we have $u \in \partial_{f^*}(x - u)$, which is equivalent to $x - u = \mathrm{prox}_{f^*}(x)$. Therefore,

$$\mathrm{prox}_f(x) + \mathrm{prox}_{f^*}(x) = u + x - u = x.$$

□

Theorem 4. *(**extended Moreau decomposition**) Let f be a closed and convex function and $\lambda > 0$. Then for any $x \in \mathbb{R}^n$, we have*

$$\mathrm{prox}_{\lambda f}(x) + \lambda \, \mathrm{prox}_{\lambda^{-1} f^*}(x/\lambda) = x.$$

Definition 2. *Let f be a closed and convex function and $\mu > 0$. The Moreau envelope of f is*

$$M_f^\mu(x) = \min_u \left\{ f(u) + \frac{1}{2\mu}\|u - x\|^2 \right\}.$$

The parameter μ is called the smoothing parameter.

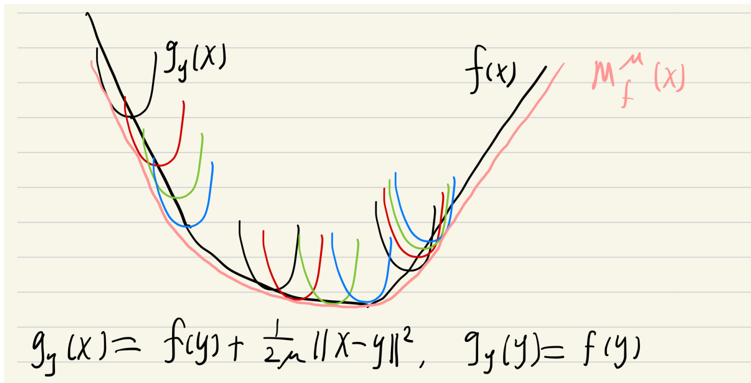

Figure 1: Moreau envelope

Properties

- $M_f^\mu(x) \le f(x)$, plot, geometric interpretation: Moreau envelope M_f^μ is an envelope underneath f that smoothifies f but may not convexifies f
- $$\nabla M_f^\mu(x) = \frac{1}{\mu}\left(x - \text{prox}_{\mu f}(x)\right)$$
- ∇M_f^μ is $\frac{1}{\mu}$-Lipschitz continuous, M_f^μ is $\frac{1}{\mu}$-smooth

$$\begin{aligned}
\|\nabla M_f^\mu(x) - \nabla M_f^\mu(y)\|^2 &= \frac{1}{\mu^2}\left\|x - \text{prox}_{\mu f}(x) - y + \text{prox}_{\mu f}(y)\right\|^2 \\
&= \frac{1}{\mu^2}\left(\|x-y\|^2 + \left\|\text{prox}_{\mu f}(x) - \text{prox}_{\mu f}(y)\right\|^2 - 2\langle \text{prox}_f(x) - \text{prox}_f(y), x - y\rangle\right) \\
&\le \frac{1}{\mu^2}\left(\|x-y\|^2 - \left\|\text{prox}_{\mu f}(x) - \text{prox}_{\mu f}(y)\right\|^2\right) \\
&\le \frac{1}{\mu^2}\|x-y\|^2
\end{aligned}$$

- M_f^μ maintians convexity if f is convex. This is because partial minimization $g(x) = \min_y f(x,y)$ preserves convexity.

3 Proximal gradient method

3.1 Composite optimization

$$\min\{\phi(x) := f(x) + h(x)\}$$

- h is closed and convex;
- f is closed and convex, $\operatorname{dom} f$ is convex, $\operatorname{dom} h \subseteq \operatorname{int}(\operatorname{dom} f)$, and f is L-smooth over $\operatorname{int}(\operatorname{dom} f)$;
- the optimal set X_* is nonempty.

3.2 Proximal gradient

Algorithm 1 Proximal gradient method

Input: Initial point $x_0 \in \operatorname{dom} h$
for $k \geq 0$ **do**
  Compute $x_{k+1} = \operatorname{prox}_h \left(x_k - h_k f'(x_k)\right)$.
end for

Theorem 5. *Functions f and h are as assumed in Subsection 3.1. Choose $\lambda \in (0, 1/L]$. Then, the proximal gradient method generates a sequence of points $\{x_k\}$ satisfying*

$$f\left(x_k\right) - f_* \leq \frac{\|x_0 - x_*\|^2}{2\lambda k}, \quad \forall k \geq 1.$$

Proof. It is easy to verify that one iteration of the proximal gradient method can be written as

$$x_{k+1} = \min_{x \in \mathbb{R}^n} \left\{ \ell_f(x; x_k) + h(x) + \frac{1}{2\lambda}\|x - x_k\|^2 \right\}.$$

Using Theorem 5 of Lecture 3 and the fact that the above objective function is $(1/\lambda)$-strongly convex, we have for every $x \in \operatorname{dom} h$,

$$\begin{aligned}
\ell_f(x; x_k) + h(x) + \frac{1}{2\lambda}\|x - x_k\|^2 &\geq \ell_f(x_{k+1}; x_k) + h(x_{k+1}) + \frac{1}{2\lambda}\|x_{k+1} - x_k\|^2 + \frac{1}{2\lambda}\|x - x_{k+1}\|^2 \\
&\geq \ell_f(x_{k+1}; x_k) + h(x_{k+1}) + \frac{L}{2}\|x_{k+1} - x_k\|^2 + \frac{1}{2\lambda}\|x - x_{k+1}\|^2 \\
&\geq f(x_{k+1}) + h(x_{k+1}) + \frac{1}{2\lambda}\|x - x_{k+1}\|^2,
\end{aligned}$$

where the second inequality is due to $\lambda \leq 1/L$ and the last inequality is due to Lemma 1(ii) of Lecture 3. It then follows from the convexity of f that

$$f(x) + h(x) + \frac{1}{2\lambda}\|x - x_k\|^2 \geq f(x_{k+1}) + h(x_{k+1}) + \frac{1}{2\lambda}\|x - x_{k+1}\|^2.$$

Taking $x = x_k$, we have

$$f(x_k) + h(x_k) \geq f(x_{k+1}) + h(x_{k+1}) + \frac{1}{2\lambda}\|x_{k+1} - x_*\|^2 \geq f(x_{k+1}) + h(x_{k+1})$$

which shows that the function value of the iterates is a nonincreasing sequence. Taking $x = x_*$, we have

$$f(x_*) + h(x_*) + \frac{1}{2\lambda}\|x_k - x_*\|^2 \geq f(x_{k+1}) + h(x_{k+1}) + \frac{1}{2\lambda}\|x_{k+1} - x_*\|^2,$$

i.e.,

$$(f+h)(x_{k+1}) - (f+h)(x_*) \leq \frac{1}{2\lambda}\|x_k - x_*\|^2 - \frac{1}{2\lambda}\|x_{k+1} - x_*\|^2.$$

Summing the above inequality and using the monotinicity of $\{(f+h)(x_k)\}$, we obtain

$$k\left[(f+h)(x_k) - (f+h)(x_*)\right] \leq \sum_{i=0}^{k-1}(f+h)(x_{i+1}) - (f+h)(x_*) \leq \frac{1}{2\lambda}\|x_0 - x_*\|^2 - \frac{1}{2\lambda}\|x_k - x_*\|^2.$$

Thus, the claim of the theorem follows. □