A Single Cut Proximal Bundle Method for Stochastic Convex Composite Optimization

Jiaming Liang

Yale University

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Joint work with Vincent Guigues (FGV) and Renato Monteiro (Georgia Tech)

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Introduction

Main problem

$$\phi_* := \min \{\phi(x) := f(x) + h(x) : x \in \mathbb{R}^n\}, \quad f(x) = \mathbb{E}_\xi[F(x,\xi)]$$

E.g., two-stage convex stochastic program

$$\min\{f_1(x) + \mathbb{E}[Q(x,\xi)] : x \in X\}$$

where $Q(x,\xi) = \min\{f_2(x,y,\xi) : g_2(x,y,\xi) \leq 0, y \in Y\}$.

An instance of the main problem with

$$h(x) = \delta_X(x), \quad F(x,\xi) = f_1(x) + Q(x,\xi).$$

Goal: SA-type algorithm based on the proximal bundle (PB) method

Assumptions

Stochastic convex composite optimization

$$\phi_* := \min \left\{ \phi(x) := f(x) + h(x) : x \in \mathbb{R}^n \right\}, \quad f(x) = \mathbb{E}_\xi[F(x,\xi)]$$

Black-box model

(A1) f is closed convex and $\operatorname{dom} f \supset \operatorname{dom} h$;

(A2) for almost every $\xi \in \Xi$, there exist a functional oracle $F(\cdot,\xi) : \operatorname{dom} h \to \mathbb{R}$ and a stochastic subgradient oracle $s(\cdot,\xi) : \operatorname{dom} h \to \mathbb{R}^n$ satisfying

$$f(x) = \mathbb{E}[F(x,\xi)], \quad f'(x) := \mathbb{E}[s(x,\xi)] \in \partial f(x);$$

(A3) for every $x \in \operatorname{dom} h$, we have $\mathbb{E}[\|s(x,\xi)\|^2] \le M^2$;

(A4) the set of optimal solutions X^* is nonempty.

Review of Deterministic PB

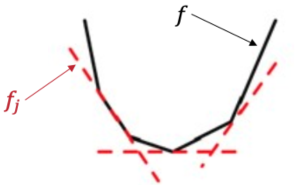
Proximal point method: constructs a sequence of proximal problems.
E.g., Chambolle-Pock for saddle point, ADMM for distributed optimization.

Approximately solve the proximal problem by an iterative process

$$x^+ \leftarrow \min_{u \in \mathbb{R}^n} \left\{ f(u) + \frac{1}{2\lambda} \|u - x^c\|^2 \right\}.$$

Recursively build up a cutting-plane model

$$f_j(u) = \max_{0 \le i \le j-1} \{ \ell_f(u; x_i) := f(x_i) + \langle f'(x_i), u - x_i \rangle \}$$



Review of Deterministic PB

Algorithm 1 PB (one cycle)

1. Construct a proximal problem

$$\min_{u \in \mathbb{R}^n} \left\{ f(u) + h(u) + \frac{1}{2\lambda}\|u - x^c\|^2 \right\};$$

2. **If** find an $(\varepsilon/2)$-solution to the current proximal problem, **then** change the prox-center; $\leftarrow$ serious

Otherwise, keep the prox-center, update the cutting-plane model and solve the prox subproblem based on the current model, i.e., $\leftarrow$ null

$$x_j = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ f_j(u) + \frac{1}{2\lambda}\|u - x^c\|^2 \right\}.$$

Cutting-plane Model in the Stochastic Setting

A straightforward fact:

$$\mathbb{E}[\max\{X, Y\}] \geq \max\{\mathbb{E}[X], \mathbb{E}[Y]\}.$$

For a fixed u,

$$\mathbb{E}[\Gamma_j(u)] \geq \max_{0 \leq i \leq j-1} \{\mathbb{E}[F(x_i, \xi_i) + \langle s(x_i, \xi_i), u - x_i \rangle]\}.$$

On the other hand,

$$\begin{aligned} &\max_{0 \leq i \leq j-1} \{\mathbb{E}[F(x_i, \xi_i) + \langle s(x_i, \xi_i), u - x_i \rangle]\} \\ = &\max_{0 \leq i \leq j-1} \{f(x_i) + \langle f'(x_i), u - x_i \rangle\} \leq f(u) \end{aligned}$$

So

$$\mathbb{E}[\Gamma_j(u)] \; ? \; f(u)$$

Other bundle models

(E1) **single cut update**[1]**:** $\Gamma^+ = \Gamma_\tau^+ := \tau\Gamma + (1-\tau)\ell_f(\cdot;x)$.

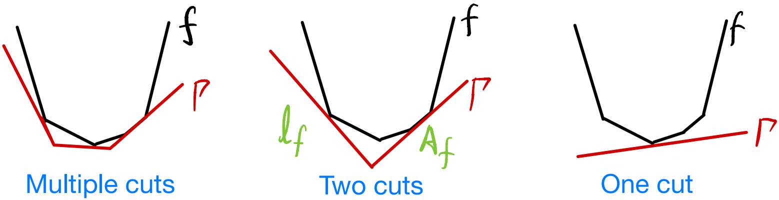
(E2) **two cuts update:** assume $\Gamma = \max\{A_f, \ell_f(\cdot;x^-)\}$ where A_f is an affine function satisfying $A_f \leq f$, set

$$\Gamma^+ = \max\{A_f^+, \ell_f(\cdot;x)\}$$

where $A_f^+ = \theta A_f + (1-\theta)\ell_f(\cdot;x^-)$.

Bundle of past information $\{(x_i, f(x_i), f'(x_i))\}$

$f \geq \Gamma$



[1] Liang and Monteiro, 2021. A unified analysis of a class of proximal bundle methods for solving hybrid convex composite optimization problems.

Single Cut Model in the Stochastic Setting

Aggregate all cuts into a single one

$$\Gamma^+(u) = \tau\Gamma(u) + (1-\tau)[F(x,\xi) + \langle s(x,\xi), u - x\rangle].$$

Since

$$\mathbb{E}[F(x,\xi) + \langle s(x,\xi), u - x\rangle] = f(x) + \langle f'(x), u - x\rangle \leq f(u),$$

we have by induction

$$\mathbb{E}[\Gamma^+(u)] \leq f(u).$$

Stochastic Composite Proximal Bundle (SCPB) Framework

1. Let $\lambda, \theta > 0$, integer $K \geq 1$, and $x_0 \in \operatorname{dom} h$ be given, and set $x_0^c = x_0$, $j = k = 1$, $j_0 = 0$, and
$$\tau = \frac{\theta K}{\theta K + 1};$$
2. Take an independent sample ξ_{j-1} of r.v. ξ, set
$$x_j^c = \begin{cases} x_{j_{k-1}}, & \text{if } j = j_{k-1} + 1, \\ x_{j-1}^c, & \text{otherwise,} \end{cases}$$
and compute
$$x_j = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ h(u) + \langle S_j, u \rangle + \frac{1}{2\lambda} \|u - x_j^c\|^2 \right\},$$
where
$$S_j := \begin{cases} s(x_{j_{k-1}}, \xi_{j_{k-1}}), & \text{if } j = j_{k-1} + 1, \\ (1-\tau) s(x_{j-1}, \xi_{j-1}) + \tau S_{j-1}, & \text{otherwise,} \end{cases}$$

SCPB Framework

3. Compute

$$y_j = \begin{cases} x_j, & \text{if } j = j_{k-1} + 1, \\ (1-\tau)x_j + \tau y_{j-1}, & \text{otherwise;} \end{cases}$$

4. Choose an integer $j_k \geq j_{k-1} + 1$, and set $\hat{y}_k = y_{j_k}$ when the k-th cycle ends;
5. if $k = K$ then **stop** and output

$$\hat{y}_K^a = \frac{1}{\lceil K/2 \rceil} \sum_{k=\lfloor K/2 \rfloor + 1}^{K} \hat{y}_k;$$

otherwise, set $k \leftarrow k + 1$ and $j \leftarrow j + 1$, and go to step 1.

Remarks on SCPB

- An aggregated single cut
- No termination criterion for a cycle

Define a cycle

$$\mathcal{C}_k := \{i_k, \ldots, j_k\}, \quad \text{where} \quad i_k := j_{k-1} + 1$$

Two ways of setting j_k:

(B1) the smallest integer $j_k \geq i_k$ and $\lambda k \tau^{j_k - i_k} \leq C$;

(B2) the smallest integer $j_k \geq i_k + 1$ and $\lambda k \tau^{j_k - i_k} t_{i_k} \leq C$.

(B1) is deterministic and (B2) is stochastic

Main Results – SCPB based on (B1)

Assume that conditions (A1)-(A4) hold and $\operatorname{dom} h$ has a finite diameter $D > 0$.

SCPB1 satisfies the following statements:

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \le \left\lceil (\theta K + 1) \ln \left(\frac{\lambda k}{C} + 1 \right) \right\rceil + 1.$$

- Convergence of SCPB1

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \le \frac{1}{K} \left(\frac{D^2}{\lambda} + \frac{6C \min\{\lambda M^2, MD\}}{\lambda} + \frac{2\lambda M^2}{\theta} \right).$$

A Practical Variant of SCPB1

Let pair (λ, K) and constant $m \geq 1$ be given, and define

$$\theta = \frac{m}{K}, \quad C = \frac{D}{6M},$$

SCPB1 satisfies the following statements:

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \leq \left\lceil (m+1) \ln\left(\frac{\lambda k}{C} + 1\right) \right\rceil + 1.$$

- Convergence of SCPB1

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + \frac{2\lambda M^2}{m}.$$

- Its expected overall iteration complexity is $\tilde{\mathcal{O}}(mK)$.

Robust Stochastic Approximation (RSA) [2]

$$x_j = \operatorname*{argmin}_{u \in X} \left\{ \langle s(x_{j-1}, \xi_{j-1}), u \rangle + \frac{1}{2\lambda} \|u - x_{j-1}\|^2 \right\} \qquad \forall j = 1, \ldots, K.$$

- Convergence of RSA

$$\mathbb{E}[\phi(x_K^a)] - \phi_* \le \frac{2D^2}{\lambda K} + 2\lambda M^2, \quad x_K^a = \frac{1}{\lceil K/2 \rceil} \sum_{j=\lfloor K/2 \rfloor + 1}^{K} x_j.$$

Taking $\lambda = \frac{\sqrt{m}D}{M\sqrt{K}}$, given $\varepsilon > 0$, to obtain $x \in \operatorname{dom} h$ such that $\mathbb{E}[\phi(x)] - \phi_* \le \varepsilon$,

- RSA has iteration complexity $\mathcal{O}\left(\frac{mM^2D^2}{\varepsilon^2}\right)$;
- SCPB1 has iteration complexity $\tilde{\mathcal{O}}\left(\frac{M^2D^2}{\varepsilon^2}\right)$.

[2] Nemirovski, Juditsky, Lan and Shapiro, 2009. Robust stochastic approximation approach to stochastic programming.

Relationship between SCPB1 and RSA

Recall (B1) the smallest integer $j_k \geq i_k$ and $\lambda k \tau^{j_k - i_k} \leq C$.
Choosing

$$C = \frac{\alpha D \sqrt{K}}{M}, \quad \lambda = \frac{\alpha D}{M\sqrt{K}},$$

then (B1) is satisfied with $j_k = i_k$, since

$$\frac{C}{\lambda k} \geq \frac{C}{\lambda K} = 1 = \tau^{j_k - i_k}.$$

In summary,

- RSA performs one iteration per cycle
- RSA $\rightarrow$ SCPB1 is analogous to Subgradient method $\rightarrow$ PB
- RSA is restricted to small stepsizes, while SCPB1 can use large ones
- SCPB1 reduces the variance and the sample complexity by m

RSA: $\mathbb{E}[\phi(x_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + 2\lambda M^2$, SCPB1: $\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + \frac{2\lambda M^2}{m}$

Main Results – SCPB based on (B2)

Recall (B2): the smallest integer $j_k \geq i_k + 1$ and $\lambda k \tau^{j_k - i_k} t_{i_k} \leq C$.

Assume that conditions (A1)-(A4) hold and $\operatorname{dom} h$ has a finite diameter $D > 0$.

SCPB2 satisfies the following statements:

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \leq \left\lceil (\theta K + 1) \ln \left(\frac{2M^2 \lambda^2 k}{C} + 1 \right) \right\rceil + 1.$$

- Convergence of SCPB2

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{1}{K} \left(\frac{3C + D^2}{\lambda} + \frac{2\lambda M^2}{\theta} + \frac{2\lambda M^2}{\theta^2 K} \right).$$

A Practical Variant of SCPB2

Let pair (λ, K) and constant $m \geq 1$ be given, and define

$$\theta = \frac{m}{K}, \quad C = \frac{D^2}{3},$$

SCPB2 satisfies the following statements:

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \leq \left\lceil (m+1) \ln \left(\frac{6M^2\lambda^2 k}{D^2} + 1 \right) \right\rceil + 1.$$

- Convergence of SCPB2

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + \frac{4\lambda M^2}{m}.$$

- Its expected overall iteration complexity is $\tilde{\mathcal{O}}(mK)$.

Test 1 – Two-stage Stochastic Program

$$\begin{cases} \min \; c^T x_1 + \mathbb{E}[Q(x_1, \xi)] \\ x_1 \in \mathbb{R}^n : x_1 \geq 0, \sum_{i=1}^n x_1(i) = 1 \end{cases}$$

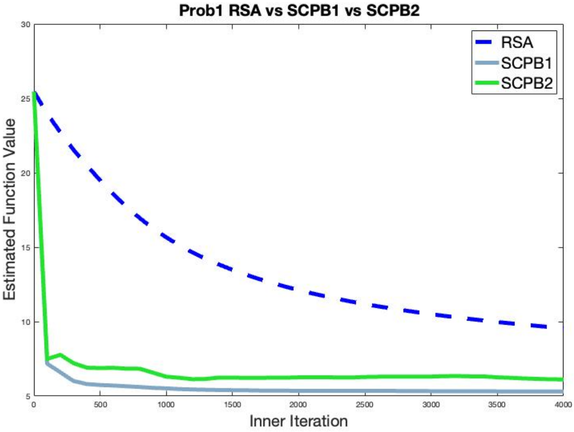
where the second stage recourse function is given by

$$Q(x_1, \xi) = \begin{cases} \min\limits_{x_2 \in \mathbb{R}^n} \; \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \left(\xi\xi^T + \lambda_0 I_{2n} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \xi^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ x_2 \geq 0, \sum\limits_{i=1}^n x_2(i) = 1. \end{cases}$$

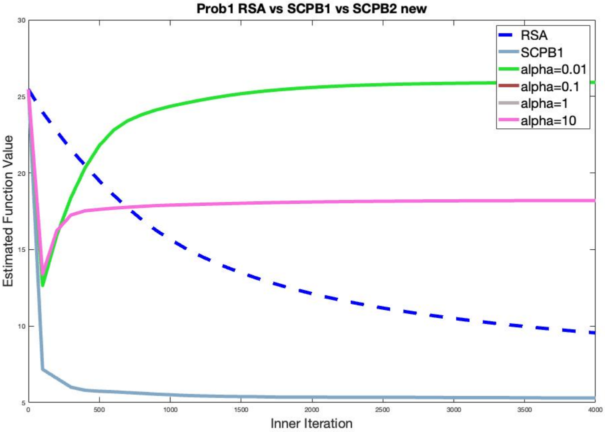
Table: $n = 50, \; N = 4000$

Statistics	RSA	SCPB1	SCPB2
λ	7.4×10^{-7}	10^{-3}	10^{-3}
Min Inner	1	9	2
Max Inner	1	52	43
Avg Inner	1	43	5

Test 1 – Two-stage Stochastic Program



Test 1 – Failure of RSA with Large Stepsize



Test 2 – Two-stage Stochastic Program

$$\begin{cases} \min \; c^T x_1 + \mathbb{E}[\mathfrak{Q}(x_1, \xi)] \\ x_1 \in \mathbb{R}^n : \|x_1 - x_0\|_2 \leq 1 \end{cases}$$

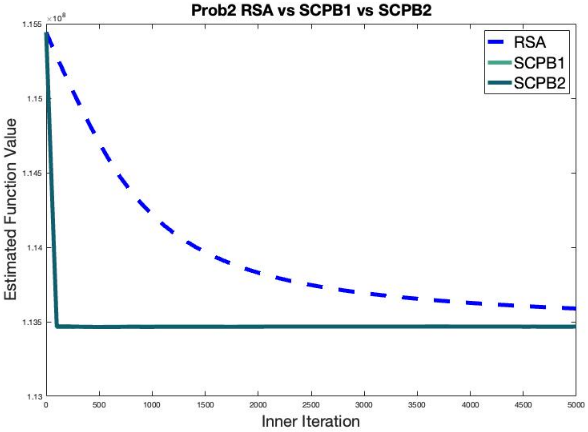
where the second stage recourse function is given by

$$Q(x_1, \xi) = \begin{cases} \min\limits_{x_2 \in \mathbb{R}^n} \; \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \left(\xi \xi^T + \lambda_0 I_{2n} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \xi^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \|x_2 - y_0\|_2^2 + \|x_1 - x_0\|_2^2 - R^2 \leq 0. \end{cases}$$

Table: $n = 50, \; N = 5000$

Statistics	RSA	SCPB1	SCPB2
λ	8.9×10^{-10}	10^{-3}	10^{-3}
Min Inner	1	71	54
Max Inner	1	109	89
Avg Inner	1	100	77

Test 2 – Two-stage Stochastic Program



Test 3 – One-stage Stochastic Program

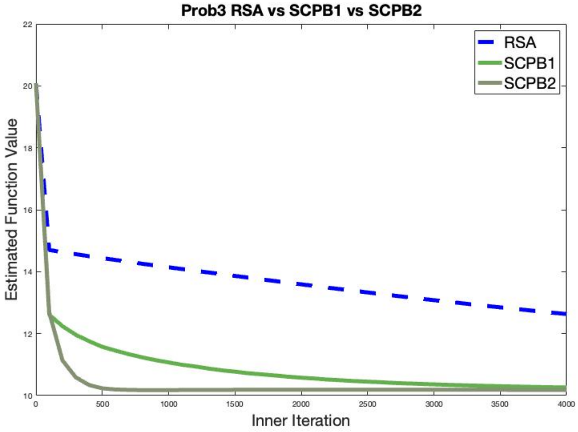
$$\min_{x \in X} \mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(\frac{i}{n} + \xi_i)x_i\right)\right]$$

where X is the unit simplex.

Table: $n = 100,\ N = 4000$

Statistics	RSA	SCPB1	SCPB2
λ	2.8×10^{-5}	10^{-3}	10^{-3}
Min Inner	1	1	2
Max Inner	1	26	6
Avg Inner	1	17	2

Test 3 – One-stage Stochastic Program



Take-away Message

- The first proximal bundle method for stochastic programming
- A single cut aggregating all past information
- Optimal complexity for large stepsizes
- Includes RSA as an instance while outperforms RSA
- Variance reduction

Reference

J. Liang, V. Guigues and R. D. C. Monteiro. A single cut proximal bundle method for stochastic convex composite optimization. ArXiv:2207.09024, 2022.

Thank you!