Early Termination of Convex QP Solvers in Mixed-Integer Model Predictive Control for Real-Time Decision Making

MIP Workshop 2021

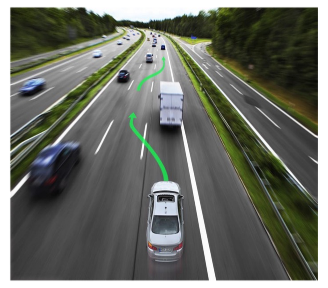
Jiaming Liang (Georgia Tech), Stefano Di Cairano, Rien Quirynen

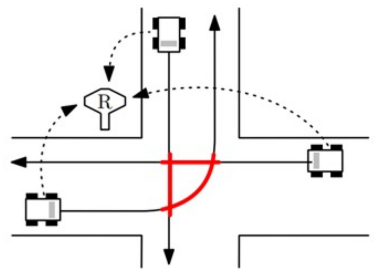
Mixed-Integer MPC for Real-Time Decision Making

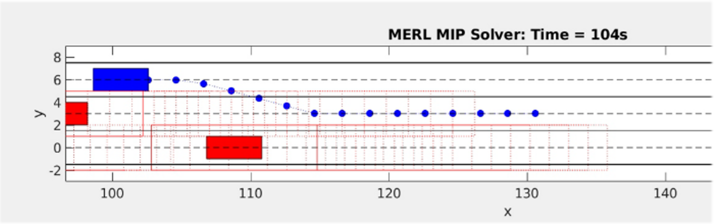
MI-MPC provides a general-purpose modeling framework for real-time decision making.

We are particularly interested in an MI-MPC formulation of a high-level motion planning task for an autonomous vehicle, including discrete decisions resulting from lane changes, static and dynamic obstacles

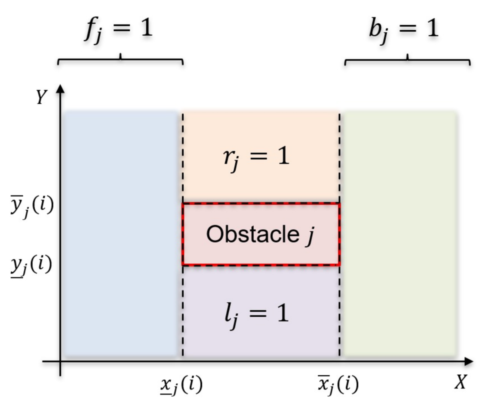
The MI-MPC framework solves an MIQP problem at every sampling time instant.







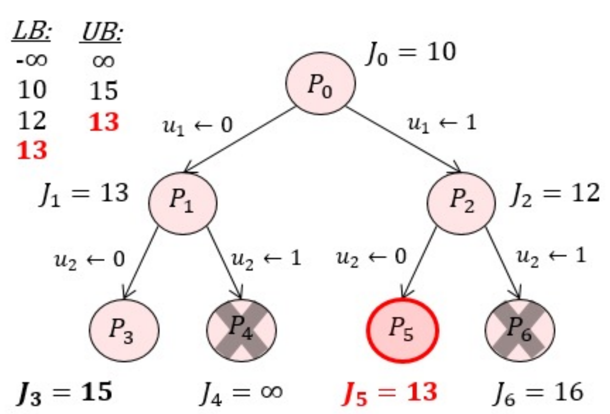
- Switches in system dynamics, e.g., contacts
- Discrete decisions, e.g., pass or stay in lane
- Quantized decisions, e.g., on/off actuation
- Disjoint constraint sets, e.g., obstacle avoidance



For example, using big-M formulation and 4 binary variables

Branch-and-Bound Algorithm for MIQP

- Convex QP relaxations solved to obtain **lower bounds (LB)**
- Each integer-feasible solution forms an **upper bound (UB)** for the MIQP solution
- A node can be pruned due to LB > UB (P_6) or infeasibility (P_4)



Early termination of QP solvers in B&B: aim to prune node without need to solve convex QP

- If dual feasible objective $\psi(\cdot) > UB$, then primal optimal objective $\phi^\star > UB$:
$$\psi(\mu,\lambda) \le \psi^\star \le \phi^\star \le \phi(\boldsymbol{x},\boldsymbol{y})$$
- Terminate the QP solver before convergence.
- Also effective in detecting primal infeasibility.

QP Formulations and Infeasible IPM Solver

Primal QP Formulation

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{y}} \quad & \phi(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{2}\boldsymbol{x}^\top Q\boldsymbol{x} + h_{\boldsymbol{x}}^\top \boldsymbol{x} + h_{\boldsymbol{y}}^\top \boldsymbol{y} \\ \text{s.t.} \quad & G_{\boldsymbol{x}}\boldsymbol{x} + G_{\boldsymbol{y}}\boldsymbol{y} \le g, \\ & F_{\boldsymbol{x}}\boldsymbol{x} + F_{\boldsymbol{y}}\boldsymbol{y} = f, \end{aligned}$$

Dual QP Formulation

$$\hat{h}(\mu,\lambda) := h_{\boldsymbol{x}} + G_{\boldsymbol{x}}^\top \mu + F_{\boldsymbol{x}}^\top \lambda,$$

$$\begin{aligned} \max_{\mu,\lambda} \quad & \psi(\mu,\lambda) := -\frac{1}{2}\|\hat{h}(\mu,\lambda)\|^2_{Q^{-1}} - \begin{bmatrix} g \\ f \end{bmatrix}^\top \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \\ \text{s.t.} \quad & G_{\boldsymbol{y}}^\top \mu + F_{\boldsymbol{y}}^\top \lambda = -h_{\boldsymbol{y}}, \\ & \mu \ge 0, \end{aligned}$$

- ***Infeasible IPM:*** Newton-type iteration
$$\begin{bmatrix} H & F^\top & G^\top \\ F & 0 & 0 \\ G & 0 & -W^k \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{z}^k \\ \Delta\lambda^k \\ \Delta\mu^k \end{bmatrix} = - \begin{bmatrix} r_{\boldsymbol{z}}^k \\ r_\lambda^k \\ \bar{r}_\mu^k \end{bmatrix}$$
- ***Problem:*** infeasible IPM iterations generally do not satisfy dual feasibility until convergence
- ***Proposed solution:*** computationally efficient projection to obtain dual feasible solution guess for early termination of infeasible IPM

Infeasible IPM: Projection to Dual Feasibility

Equality-constrained optimization for minimum-norm projection on constraint

$$\min_{\Delta\lambda,\Delta\mu} \quad \frac{1}{2}\|\Delta\lambda\|^2 + \frac{1}{2}\|\Delta\mu\|^2, \quad \text{s.t.} \quad F_{\boldsymbol{y}}^\top \Delta\lambda + G_{\boldsymbol{y}}^\top \Delta\mu = -r_{\boldsymbol{y}}^k$$

But projection does not guarantee nonnegativity of Lagrange multipliers, i.e., $\mu \geq 0$

Proposed approach: modified optimization problem for projection on constraint

Solve equality-constrained optimization problem to compute **projection step**

$$\begin{aligned} \min_{\Delta\boldsymbol{x},\Delta\lambda,\Delta\mu} \quad & \frac{1}{2}\|\Delta\boldsymbol{x}\|^2_Q + \frac{1}{2}\|\Delta\lambda\|^2_{\epsilon_{\text{dual}}} + \boxed{\frac{1}{2}\|\Delta\mu\|^2_{W^k}} \\ \text{s.t.} \quad & \begin{bmatrix} Q \\ 0 \end{bmatrix} \Delta\boldsymbol{x} + F^\top\Delta\lambda + G^\top\Delta\mu = -\begin{bmatrix} 0 \\ r_{\boldsymbol{y}}^k \end{bmatrix}, \end{aligned}$$

$$W^k = \operatorname{diag}(w^k) \text{ and } w_i^k = \frac{s_i^k}{\mu_i^k} > 0$$
$$\Downarrow$$
$$\|\Delta\mu\|^2_{W^k} = \textstyle\sum_i \left(\frac{s_i^k}{\mu_i^k}\Delta\mu_i^2\right)$$

Indirectly enforces **positivity** constraints $\mu \geq 0$

Active inequality constraints:

$s_i \to 0$ and $\mu_i > 0 \Rightarrow$ $w_i = \frac{s_i}{\mu_i} \to 0$ (small penalty)

Inactive inequality constraints:

$\mu_i \to 0$ and $s_i > 0 \Rightarrow$ $w_i = \frac{s_i}{\mu_i} \to \infty$ (large penalty)

$$\equiv$$

is **equivalent** to solving KKT system to compute Newton-type search direction

$$\begin{bmatrix} H & F^\top & G^\top \\ F & -\epsilon_{\text{dual}} I & 0 \\ G & 0 & -W^k \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{z} \\ \Delta\lambda \\ \Delta\mu \end{bmatrix} = - \begin{bmatrix} \begin{bmatrix} 0 \\ r_{\boldsymbol{y}}^k \end{bmatrix} \\ 0 \\ 0 \end{bmatrix},$$

Projection corresponds to IPM iteration with different right-hand side: obtain dual feasibility while maintaining optimality conditions

$$\equiv$$

is **equivalent** to solving **reduced** KKT system to compute search direction Δz

$$\boxed{H + \frac{1}{\epsilon_{\text{dual}}}F^\top F + G^\top W^{k^{-1}} G}\,\Delta z = -\begin{bmatrix} 0 \\ r_y^k \end{bmatrix},$$

which is used to compute projection step for Lagrange multipliers $(\Delta\mu^k, \Delta\lambda^k)$:

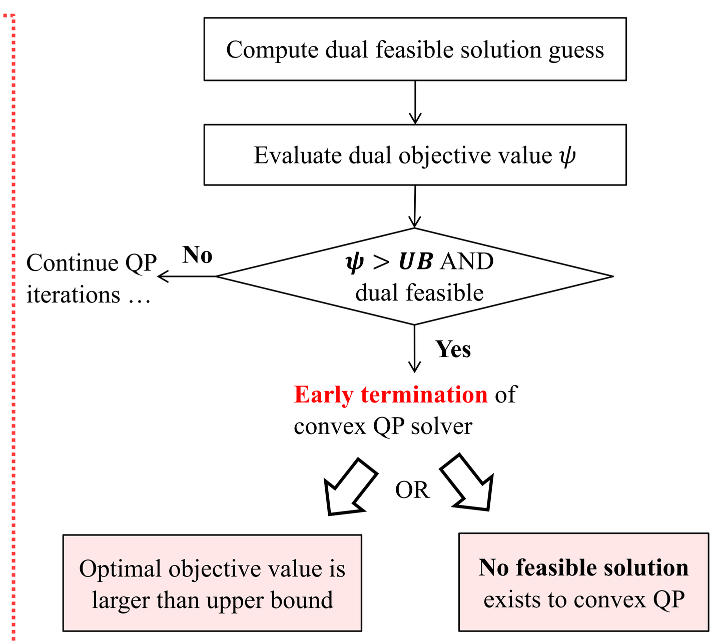
$$\Delta\lambda = \frac{1}{\epsilon_{\text{dual}}}F\Delta z, \quad \Delta\mu = W^{k^{-1}} G\Delta z$$

We can **reuse** the KKT matrix factorization between IPM iterations and projection steps for **computational efficiency**

Early Termination of IPM: Infeasibility detection

- **Certificate** of primal infeasibility (i.e., unboundedness of dual) the following set of equations is strictly infeasible
$$Gz < g, \quad Fz = f,$$
if and only if there exists a pair $(\tilde{\mu}, \tilde{\lambda})$ such that **(Farkas' lemma)**
$$G^\top\tilde{\mu} + F^\top\tilde{\lambda} = 0, \quad g^\top\tilde{\mu} + f^\top\tilde{\lambda} < 0, \quad \tilde{\mu} > 0$$
- Instead, our proposed early termination technique can be used for infeasibility detection and requires limited computational cost (projection based on reuse of KKT matrix factorization).
- Intuition behind using **early termination for infeasibility detection:**

Proposition 4.3: If the sequence of IPM iterates $\{(z^k,\mu^k,\lambda^k,s^k)\}$ satisfy $\mu^{k^\top}s^k \le \mu^{0^\top}s^0$ and $\|\mu^k\| \to \infty$, then the dual objective $\psi(\mu^k,\lambda^k) \to \infty$.



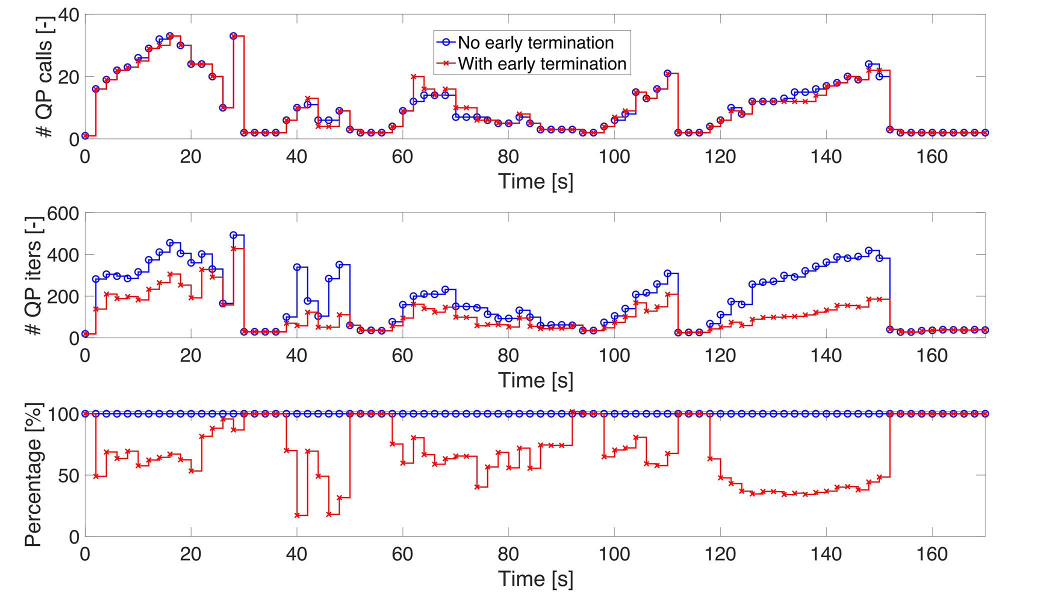
Algorithm 1 Early termination for IPM in B&B method.

```
Input: Warm start {(z^0, μ^0, λ^0, s^0)}, tol, and UB.
while max{τ^k, ||r^k||} > tol do
    if ψ(μ^k, λ^k) > UB & dual_feasible then
        break while loop.                          ▷ Early termination
    else if ψ(μ^k, λ^k) > UB then
        Compute projection step (Δμ, Δλ) in (13).
        μ ← μ^k + Δμ, λ ← λ^k + Δλ, and
        r_y ← F_y^T λ + G_y^T μ + h_y.
        if μ > 0 & ||r_y|| < tol then
            μ^k ← μ, λ^k ← λ, r_y^k ← r_y, and
            dual_feasible ← 1.
            if ψ(μ^k, λ^k) > UB then
                break while loop.                  ▷ Early termination
            end if
        end if
    end if
    Perform an IPM iteration (8), e.g., see [18].
end while
```

Simulation Results: Real-Time Vehicle Decision Making

QP Calls and Total Iterations

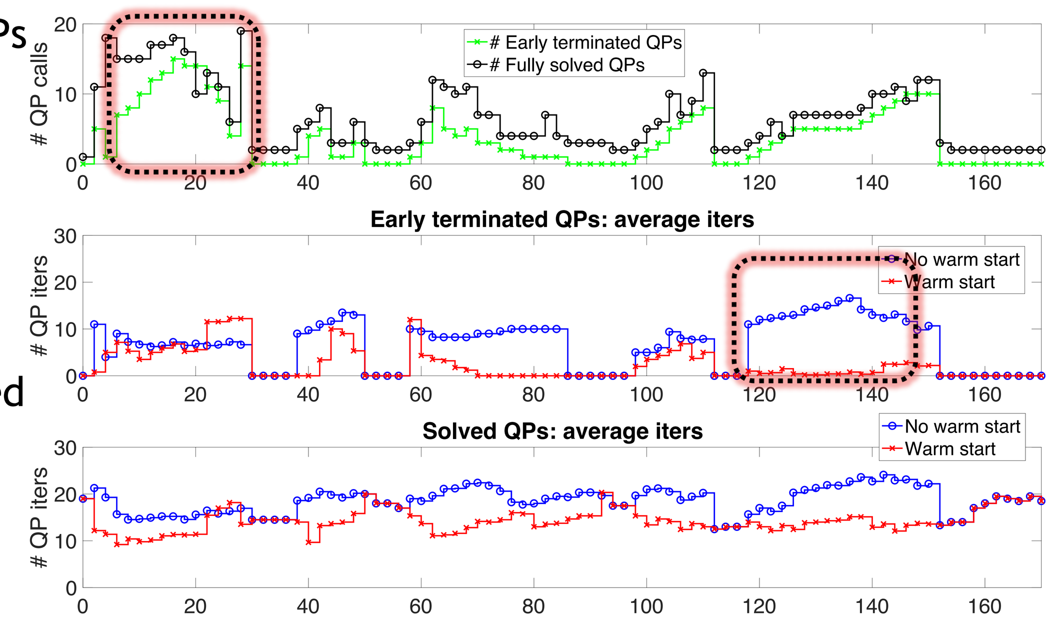
blue: No early termination
red: Early termination



- Early termination **reduces** total number **of QP iterations by 42%.**

Early Terminated versus Fully Solved QPs

green: Early terminated QPs
black: Fully solved QPs
blue: No warm start
red: Warm start



- Early termination happens often, **36%** of QPs are early terminated
- Early termination benefits from warm starting IPM and takes less QP iterations.

Infeasibility Detection

Number of IPM Iterations for Certificate Versus Early Termination With and Without Warm Starting

	QP # 1	QP # 2	QP # 3
Certificate of primal infeasibility	40	45	38
Early termination: cold started	10	12	10
Early termination: warm started	0	0	11

- Early termination requires **considerably less IPM iterations** than the computation of a certificate of infeasibility.
- **Warm starting** can reduce the number of IPM iterations further and it can lead to immediate termination, i.e., termination at 0 iterations.

Conclusions

An efficient early termination strategy based on a projection step tailored to IPMs, in order to reduce the computational cost within B&B method in solving MI-MPC.

Early termination of QP solvers in MI-MPC works well in

- terminating QPs whose objective value > UB;
- detecting infeasible QPs.

Early termination is performed by using Newton-type IPM iterations

- reuses KKT matrix factorizations for computational efficiency;
- intuitively guarantees the inequality constraint, i.e., positivity in μ;
- projection also makes progress towards convergence.